MEI PURE 5 JANUARY 2002 QUESTION 3

3 i) Prove $\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right)$

Let $y = \operatorname{arsinh} x$

$$\Rightarrow \sinh y = x$$

$$\tfrac{1}{2}(e^y - e^{-y}) = x$$

$$e^y - e^{-y} = 2x$$

$$e^{2y} - 1 = 2xe^y$$

$$e^{2y} - 2xe^y - 1 = 0$$

$$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$\Rightarrow e^y = \frac{2x \pm 2\sqrt{x^2+1}}{2}$$

$$\Rightarrow e^y = x \pm \sqrt{x^2+1}$$

$$\Rightarrow e^y = x + \sqrt{x^2+1}$$

since $e^y > 0$

$$\Rightarrow y = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\therefore \operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right)$$

3 ii) $$\int_0^2 \frac{1}{\sqrt{3x^2+4}}\,dx$$

$$= \frac{1}{\sqrt{3}} \int_0^2 \frac{1}{\sqrt{x^2+\frac{4}{3}}}\,dx$$

Standard integral

$$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \operatorname{arsinh}\left(\frac{x}{a}\right) = \ln\left(x + \sqrt{x^2+a^2}\right)$$

$$= \frac{1}{\sqrt{3}} \int_0^2 \frac{1}{\sqrt{x^2+\left(\frac{2}{\sqrt{3}}\right)^2}}\,dx$$

$$= \frac{1}{\sqrt{3}}\left[\ln\left(x + \sqrt{x^2+\tfrac{4}{3}}\right)\right]_0^2$$

$$= \frac{1}{\sqrt{3}}\left[\ln\left(2 + \sqrt{4+\tfrac{4}{3}}\right) - \ln\left(0 + \sqrt{\tfrac{4}{3}}\right)\right]$$

$$= \frac{1}{\sqrt{3}}\left[\ln\left(2 + \sqrt{\tfrac{16}{3}}\right) - \ln\left(\frac{2}{\sqrt{3}}\right)\right]$$

$$= \frac{1}{\sqrt{3}}\left[\ln\left(\frac{2\sqrt{3}+4}{\sqrt{3}}\right) - \ln\left(\frac{2}{\sqrt{3}}\right)\right]$$

$$= \frac{1}{\sqrt{3}} \ln\left(\frac{\frac{2\sqrt{3}+4}{\sqrt{3}}}{\frac{2}{\sqrt{3}}}\right)$$

$$= \frac{1}{\sqrt{3}} \ln\left(\frac{2\sqrt{3}+4}{\sqrt{3}} \times \frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{\sqrt{3}} \ln\left(\frac{2\sqrt{3}+4}{2}\right)$$

$$= \frac{1}{\sqrt{3}} \ln\left(\sqrt{3}+2\right)$$

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3iii)
$$\int_0^2 \frac{1}{3x^2+4}\,dx$$

$$= \frac{1}{3}\int_0^2 \frac{1}{x^2+\frac{4}{3}}\,dx$$

Standard integral

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{1}{3}\int_0^2 \frac{1}{x^2+\left(\frac{2}{\sqrt{3}}\right)^2}\,dx$$

$$= \frac{1}{3}\,\frac{1}{\frac{2}{\sqrt{3}}}\left[\tan^{-1}\left(\frac{x}{2/\sqrt{3}}\right)\right]_0^2$$

$$= \frac{\sqrt{3}}{6}\left[\tan^{-1}\left(\frac{\sqrt{3}x}{2}\right)\right]_0^2$$

$$= \frac{\sqrt{3}}{6}\left[\tan^{-1}\sqrt{3} - \tan^{-1}0\right]$$

$$= \frac{\sqrt{3}}{6}\left[\frac{\pi}{3} - 0\right]$$

$$= \frac{\sqrt{3}\pi}{18} \quad \text{or} \quad \frac{\pi}{6\sqrt{3}}$$

3iv)
$$\int_0^2 \frac{1}{(3x^2+4)^{3/2}}\,dx$$

Let $x\sqrt{3} = 2\tan\theta$

$$\sqrt{3}\frac{dx}{d\theta} = 2\sec^2\theta$$

$$\frac{dx}{d\theta} = \frac{2}{\sqrt{3}}\sec^2\theta$$

$$dx = \frac{2}{\sqrt{3}}\sec^2\theta\,d\theta$$

when $x = 2$, $\tan\theta = \sqrt{3}$ $\Rightarrow \theta = \frac{\pi}{3}$

when $x = 0$ $\tan\theta = 0$ $\Rightarrow \theta = 0$

$$\int_0^{\frac{\pi}{3}} \frac{1}{(4\tan^2\theta+4)^{3/2}}\cdot\frac{2\sec^2\theta\,d\theta}{\sqrt{3}}$$

$(1+\tan^2\theta = \sec^2\theta)$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{4^{3/2}(\tan^2\theta+1)^{3/2}}\,\frac{2}{\sqrt{3}}\sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{8\sec^3\theta}\cdot\frac{2}{\sqrt{3}}\sec^2\theta\,d\theta$$

$$= \frac{1}{4\sqrt{3}}\int_0^{\frac{\pi}{3}} \frac{1}{\sec\theta}\,d\theta = \frac{1}{4\sqrt{3}}\int_0^{\frac{\pi}{3}} \cos\theta\,d\theta$$

$$= \frac{1}{4\sqrt{3}}\left[\sin\theta\right]_0^{\frac{\pi}{3}} = \frac{1}{4\sqrt{3}}\left[\frac{\sqrt{3}}{2} - 0\right]$$

$$= \frac{1}{8}$$

MEI PURE 5 MAY 2002 QUESTION 3

3a i) $y = \cosh 2x - 3\sinh x$

$\frac{dy}{dx} = 2\sinh 2x - 3\cosh x$

At st. pt. $\frac{dy}{dx} = 0$

$\Rightarrow 2\sinh 2x - 3\cosh x = 0$

$\Rightarrow 4\sinh x\cosh x - 3\cosh x = 0$

$\Rightarrow \cosh x(4\sinh x - 3) = 0$

$\Rightarrow \cosh x = 0$ (impossible)

or $4\sinh x - 3 = 0$

$4\sinh x = 3$

$\sinh x = \frac{3}{4}$

$x = \operatorname{arsinh}\left(\frac{3}{4}\right)$

$x = \ln\left(\frac{3}{4} + \sqrt{\frac{9}{16} + 1}\right)$

(using $\operatorname{arsinh} x = \ln(x + \sqrt{x^2+1})$)

$x = \ln\left(\frac{3}{4} + \sqrt{\frac{25}{16}}\right)$

$x = \ln\left(\frac{3}{4} + \frac{5}{4}\right) = \ln 2$

When $x = \ln 2$

$y = \cosh(2\ln 2) - 3\sinh(\ln 2)$

$y = \cosh(\ln 4) - 3\sinh(\ln 2)$

$y = \frac{1}{2}\left(e^{\ln 4} + e^{-\ln 4}\right) - \frac{3}{2}\left(e^{\ln 2} - e^{-\ln 2}\right)$

$y = \frac{1}{2}\left(4 + \frac{1}{4}\right) - \frac{3}{2}\left(2 - \frac{1}{2}\right)$

$y = \frac{17}{8} - \frac{9}{4} = \frac{17}{8} - \frac{18}{8}$

$y = -\frac{1}{8}$

$\therefore \left(\ln 2, -\frac{1}{8}\right)$ is only st. pt.

3a ii) $\int_0^{\ln 2} (\cosh 2x - 3\sinh x)\,dx$

$= \left[\frac{1}{2}\sinh 2x - 3\cosh x\right]_0^{\ln 2}$

$= \left(\frac{1}{2}\sinh(2\ln 2) - 3\cosh(\ln 2)\right) - \left(\frac{1}{2}\sinh 0 - 3\cosh 0\right)$

$= \frac{1}{2}\cdot\frac{1}{2}\left(e^{\ln 4} - e^{-\ln 4}\right) - 3\cdot\frac{1}{2}\left(e^{\ln 2} + e^{-\ln 2}\right) - (0 - 3)$

$= \frac{1}{4}\left(4 - \frac{1}{4}\right) - \frac{3}{2}\left(2 + \frac{1}{2}\right) + 3$

$= \frac{15}{16} - \frac{15}{4} + 3$

$= \frac{15}{16} - \frac{60}{16} + \frac{48}{16}$

$= \frac{3}{16}$

MEI PURE 5 MAY 2002 QUESTION 3

3b i) $f(x) = \arctan(1+x)$

Let $y = \arctan(1+x)$

$\Rightarrow \tan y = 1+x$

$\Rightarrow \sec^2 y \dfrac{dy}{dx} = 1$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\sec^2 y}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{1+\tan^2 y}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{1+(1+x)^2}$

$\Rightarrow f'(x) = \dfrac{1}{2+2x+x^2}$

$$f''(x) = \frac{(x^2+2x+2)0 - 1(2x+2)}{(2+2x+x^2)^2}$$

$$f''(x) = \frac{-2x-2}{(2+2x+x^2)^2}$$

3b ii) Let $f(x) = \arctan(1+x)$

Then $f(x) \approx f(0) + x f'(0) + \dfrac{x^2}{2!} f''(0)$

$f(0) = \arctan(1) = \dfrac{\pi}{4}$

$f'(0) = \dfrac{1}{2+0+0} = \dfrac{1}{2}$

$f''(0) = \dfrac{-2}{2^2} = -\dfrac{1}{2}$

$f(x) \approx \dfrac{\pi}{4} + \dfrac{x}{2} - \dfrac{x^2}{2!} \cdot \dfrac{1}{2}$

$\arctan(1+x) \approx \dfrac{\pi}{4} + \dfrac{x}{2} - \dfrac{x^2}{4}$

3b iii)

$$\int_0^{0.4} \arctan(1+x^2)\,dx$$

$$\approx \int_0^{0.4} \left(\frac{\pi}{4} + \frac{x^2}{2} - \frac{x^4}{4}\right) dx$$

$$= \left[\frac{\pi}{4}x + \frac{x^3}{6} - \frac{x^5}{20}\right]_0^{0.4}$$

$$= \left(\frac{0.4\pi}{4} + \frac{0.4^3}{6} - \frac{0.4^5}{20}\right) - (0+0-0)$$

$= 0.32431$

$= 0.324$ to 3 d.p.

MEI PURE 5 JANUARY 2003 QUESTION 1

1 i)

$$\cosh x = \frac{1}{2}\left(e^{x}+e^{-x}\right)$$

$$2\cosh^2 x - 1 = 2\left(\frac{1}{2}\left(e^{x}+e^{-x}\right)\right)^2 - 1$$

$$= 2\left(\frac{1}{4}\left(e^{2x}+e^{-2x}+2\right)\right) - 1$$

$$= \frac{1}{2}\left(e^{2x}+e^{-2x}\right)+1-1$$

$$= \cosh 2x$$

1 ii)

$$\cosh 2x - 10\cosh x + 13 = 0$$

$$2\cosh^2 x - 1 - 10\cosh x + 13 = 0$$

$$2\cosh^2 x - 10\cosh x + 12 = 0$$

$$\cosh^2 x - 5\cosh x + 6 = 0$$

$$(\cosh x - 2)(\cosh x - 3) = 0$$

$$\Rightarrow \cosh x = 2 \quad \text{or} \quad \cosh x = 3$$

$$\Rightarrow x = \operatorname{arcosh} 2 \quad \text{or} \quad x = \operatorname{arcosh} 3$$

using $\operatorname{arcosh} x = \pm\ln\left(x+\sqrt{x^2-1}\right)$

$$x = \pm\ln\left(2+\sqrt{2^2-1}\right)$$

or $x = \pm\ln\left(3+\sqrt{3^2-1}\right)$

$$x = \pm\ln\left(2+\sqrt{3}\right)$$

or $x = \pm\ln\left(3+\sqrt{8}\right)$

1 iii)

$$y = \cosh 2x - 10\cosh x + 13$$

$$\frac{dy}{dx} = 2\sinh 2x - 10\sinh x$$

At st.pt. $\frac{dy}{dx} = 0$

$$\Rightarrow 2\sinh 2x - 10\sinh x = 0$$

$$\Rightarrow \sinh 2x - 5\sinh x = 0$$

$$\Rightarrow 2\sinh x\cosh x - 5\sinh x = 0$$

$$\Rightarrow \sinh x\left(2\cosh x - 5\right) = 0$$

$$\Rightarrow \sinh x = 0 \Rightarrow x = 0$$

or $2\cosh x - 5 = 0$

$$\cosh x = \frac{5}{2}$$

$$x = \pm\ln\left(\frac{5}{2}+\sqrt{\frac{25}{4}-1}\right)$$

$$x = \pm\ln\left(\frac{5}{2}+\frac{\sqrt{21}}{2}\right)$$

$\therefore$ 3 st.pts. at

$$x = 0, \quad x = \pm\ln\left(\frac{5}{2}+\frac{\sqrt{21}}{2}\right)$$

when $x = 0$

$$y = \cosh 0 - 10\cosh 0 + 13$$

$$y = 1 - 10 + 13$$

$$\underline{y = 4}$$

MEI PURE 5 JANUARY 2003 QUESTION 1

1 iii) cont.

when $x = \pm\ln\left(\frac{5}{2} + \frac{\sqrt{21}}{2}\right)$

$$\cosh x = \frac{5}{2}$$

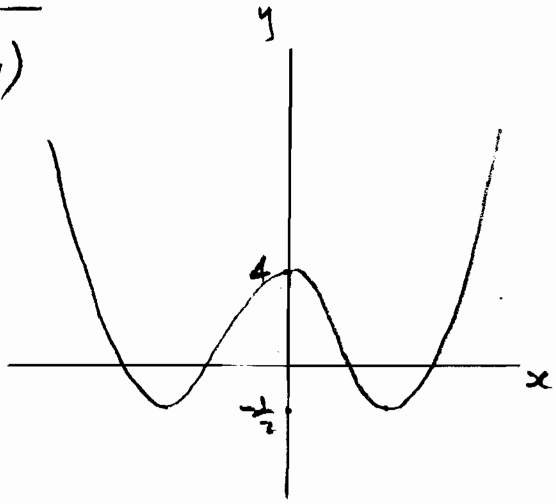
$$\cosh 2x = 2\cosh^2 x - 1$$
$$= 2\left(\frac{25}{4}\right) - 1$$
$$= \frac{23}{2}$$

$$\therefore y = \cosh 2x - 10\cosh x + 13$$
$$= \frac{23}{2} - 10 \times \frac{5}{2} + 13$$
$$= \frac{23}{2} - 25 + 13$$
$$= -\frac{1}{2}$$

st pts.

$$(0, 4)$$
$$\left(\ln\left(\frac{5}{2} + \frac{\sqrt{21}}{2}\right), -\frac{1}{2}\right)$$
$$\left(-\ln\left(\frac{5}{2} + \frac{\sqrt{21}}{2}\right), -\frac{1}{2}\right)$$

1 iv)



Note $\cosh 2x$ is much bigger than $10\cosh x$ as x increases towards ∞ or decreases towards $-\infty$.

MEI PURE 5 JUNE 2003 QUESTION 2

2 i) Prove $\operatorname{arsinh} x = \ln\left(x+\sqrt{x^2+1}\right)$

Let $y = \operatorname{arsinh} x$

$\Rightarrow \sinh y = x$

$\Rightarrow \frac{1}{2}(e^y - e^{-y}) = x$

$\Rightarrow e^y - e^{-y} = 2x$

$\Rightarrow e^{2y} - 1 = 2xe^y$

$\Rightarrow e^{2y} - 2xe^y - 1 = 0$

$\Rightarrow e^y = \dfrac{2x \pm \sqrt{4x^2+4}}{2}$

$\Rightarrow e^y = \dfrac{2x \pm 2\sqrt{x^2+1}}{2}$

$\Rightarrow e^y = x \pm \sqrt{x^2+1}$

$\Rightarrow e^y = x + \sqrt{x^2+1}$

since $e^y > 0$

$\Rightarrow y = \ln\left(x+\sqrt{x^2+1}\right)$

$\Rightarrow \operatorname{arsinh} x = \ln\left(x+\sqrt{x^2+1}\right)$

2ii) $\displaystyle\int_0^2 \frac{1}{\sqrt{4x^2+9}}\,dx$

$= \dfrac{1}{2}\displaystyle\int_0^2 \frac{1}{\sqrt{x^2+\frac{9}{4}}}\,dx$

Standard integral

$\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \operatorname{arsinh}\left(\frac{x}{a}\right)$

or $\ln\left(x+\sqrt{x^2+a^2}\right)$

$= \dfrac{1}{2}\displaystyle\int_0^2 \frac{1}{\sqrt{x^2+\left(\frac{3}{2}\right)^2}}\,dx$

$= \dfrac{1}{2}\left[\operatorname{arsinh}\left(\dfrac{x}{3/2}\right)\right]_0^2$

$= \dfrac{1}{2}\left[\operatorname{arsinh}\left(\dfrac{2x}{3}\right)\right]_0^2$

$= \dfrac{1}{2}\left[\operatorname{arsinh}\left(\dfrac{4}{3}\right) - \operatorname{arsinh} 0\right]$

$= \dfrac{1}{2}\operatorname{arsinh}\left(\dfrac{4}{3}\right)$

$= \dfrac{1}{2}\ln\left(\dfrac{4}{3} + \sqrt{\dfrac{16}{9}+1}\right)$

$= \dfrac{1}{2}\ln\left(\dfrac{4}{3}+\dfrac{5}{3}\right)$

$= \dfrac{1}{2}\ln 3$

2iii) $f(x) = \operatorname{arsinh}\left(\frac{4}{3}+x\right)$

Let $y = \operatorname{arsinh}\left(\frac{4}{3}+x\right)$

$\Rightarrow \sinh y = \frac{4}{3}+x$

$\Rightarrow \cosh y \dfrac{dy}{dx} = 1$

$\dfrac{dy}{dx} = \dfrac{1}{\cosh y}$

Now $\cosh^2 y - \sinh^2 y = 1$

so $\cosh y = \sqrt{1+\sinh^2 y}$

$\cosh y = \sqrt{1+\left(\frac{4}{3}+x\right)^2}$

MEI PURE 5 JUNE 2003 QUESTION 2

2 iii) cont

$$f'(x) = \frac{dy}{dx} = \frac{1}{\sqrt{1+\left(\frac{4}{3}+x\right)^2}}$$

or $$f'(x) = \left(1+\left(\tfrac{4}{3}+x\right)^2\right)^{-\frac{1}{2}}$$

$$\Rightarrow f''(x) = -\frac{1}{2}\left(1+\left(\tfrac{4}{3}+x\right)^2\right)^{-3/2} \times 2\left(\tfrac{4}{3}+x\right)$$

$$f''x = -\frac{\left(\frac{4}{3}+x\right)}{\left(1+\left(\frac{4}{3}+x\right)^2\right)^{3/2}}$$

2iv) Let $f(x) = \operatorname{arsinh}\left(\frac{4}{3}+x\right)$

$$f(0) = \operatorname{arsinh}\left(\tfrac{4}{3}\right) = \ln\left(\tfrac{4}{3}+\sqrt{\tfrac{16}{9}+1}\right) = \ln\left(\tfrac{4}{3}+\tfrac{5}{3}\right) = \ln 3$$

$$f'(0) = \frac{1}{\sqrt{1+\frac{16}{9}}} = \frac{1}{5/3}$$

$$f'(0) = \frac{3}{5}$$

$$f''(0) = \frac{-\frac{4}{3}}{\left(\frac{25}{9}\right)^{3/2}}$$

$$f''(0) = \frac{-\frac{4}{3}}{\frac{125}{27}}$$

$$f''(0) = -\frac{4}{3}\times\frac{27}{125} = -\frac{36}{125}$$

$$f(x) \approx f(0) + f'(0)x + f''(0)\frac{x^2}{2!}$$

$$\therefore \operatorname{arsinh}\left(\tfrac{4}{3}+x\right) = f(x) \approx \ln 3 + \frac{3x}{5} - \frac{36}{125}\cdot\frac{x^2}{2}$$

$$= \ln 3 + \frac{3}{5}x - \frac{18}{125}x^2$$

2v) $$\int_{-h}^{h} x \operatorname{arsinh}\left(\tfrac{4}{3}+x\right) dx$$

$$\approx \int_{-h}^{h}\left(x\ln 3 + \frac{3}{5}x^2 - \frac{18}{125}x^3\right)dx$$

$$= \left[\frac{x^2}{2}\ln 3 + \frac{x^3}{5} - \frac{18x^4}{500}\right]_{-h}^{h}$$

$$= \left(\frac{h^2}{2}\ln 3 + \frac{h^3}{5} - \frac{18h^4}{500}\right) - \left(\frac{(-h)^2}{2}\ln 3 + \frac{(-h)^3}{5} - \frac{18(-h)^4}{500}\right)$$

$$= \frac{h^3}{5} + \frac{h^3}{5} = \frac{2h^3}{5}$$

(other terms cancel out)

$$\therefore \int_{-h}^{h} x\operatorname{arsinh}\left(\tfrac{4}{3}+x\right)dx \approx \frac{2h^3}{5}$$

when h is small