MEI PURE 5 JANUARY 2004 QUESTION 3

3 i)

A) $1 + 2\sinh^2 u$

$= 1 + 2\left[\frac{1}{2}(e^u - e^{-u})\right]^2$

$= 1 + \frac{1}{2}\left(e^{2u} + e^{-2u} - 2\right)$

$= 1 + \frac{1}{2}\left(e^{2u} + e^{-2u}\right) - 1$

$= \cosh 2u$

B) Let $y = \operatorname{arcosh} x$

$\Rightarrow \cosh y = x$

$\frac{1}{2}(e^y + e^{-y}) = x$

$e^y + e^{-y} = 2x$

$e^{2y} + 1 = 2xe^y$

$e^{2y} - 2xe^{2y} + 1 = 0$

$\Rightarrow e^y = \dfrac{2x \pm \sqrt{4x^2 - 4}}{2}$

$\Rightarrow e^y = x \pm \sqrt{x^2 - 1}$

$\Rightarrow y = \ln\left(x \pm \sqrt{x^2 - 1}\right)$

Restricting $y > 0$ for 1:1 mappings

$y = \operatorname{arcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right)$

3 ii) $\displaystyle\int_0^{\ln 5} \sinh^2 u \, du$

$= \displaystyle\int_0^{\ln 5} \left(\frac{1}{2}(e^u - e^{-u})\right)^2 du$

$= \displaystyle\int_0^{\ln 5} \frac{1}{4}\left(e^{2u} + e^{-2u} - 2\right) du$

$= \frac{1}{4}\left[\frac{1}{2}e^{2u} - \frac{1}{2}e^{-2u} - 2u\right]_0^{\ln 5}$

$= \frac{1}{4}\left[\left(\frac{1}{2}e^{2\ln 5} - \frac{1}{2}e^{-2\ln 5} - 2\ln 5\right) - \left(\frac{1}{2} - \frac{1}{2} - 0\right)\right]$

$= \frac{1}{4}\left[\frac{1}{2} \times 25 - \frac{1}{2} \times \frac{1}{25} - 2\ln 5\right]$

$= \frac{25}{8} - \frac{1}{200} - \frac{1}{2}\ln 5$

$= \frac{625}{200} - \frac{1}{200} - \frac{1}{2}\ln 5$

$= \frac{624}{200} - \frac{1}{2}\ln 5$

$= \frac{78}{25} - \frac{1}{2}\ln 5$

3 iii) $\displaystyle\int_{0.625}^{1.3} \frac{1}{\sqrt{4x^2 - 1}} \, dx$

$= \frac{1}{2}\displaystyle\int_{0.625}^{1.3} \frac{1}{\sqrt{x^2 - \frac{1}{4}}} \, dx$

$= \frac{1}{2}\left[\ln\left(x + \sqrt{x^2 - \frac{1}{4}}\right)\right]_{0.625}^{1.3}$

$= \frac{1}{2}\left[\ln(1.3 + 1.2) - \ln(0.625 + 0.375)\right]$

MEI PURE 5 JANUARY 2004 QUESTION 3

3iii) cont)

$$= \frac{1}{2}\left[\ln 2.5 - \ln 1\right]$$

$$= \frac{1}{2}\ln\frac{5}{2} = \frac{1}{2}\left[\ln 5 - \ln 2\right]$$

3iv)

$$\int_{0.5}^{1.3} \sqrt{4x^2-1}\, dx$$

Now $\cosh^2 u - \sinh^2 u = 1$

$\therefore \cosh^2 u - 1 = \sinh^2 u$

so use substitution

$$\cosh u = 2x$$

$$\Rightarrow \sinh u \frac{du}{dx} = 2$$

$$\sinh u\, du = 2\, dx$$

$$\frac{1}{2}\sinh u\, du = dx$$

when $x = 1.3$
$u = \operatorname{arcosh} 2.6$

when $x = 0.5$
$u = \operatorname{arcosh} 1$

$$\int_{\operatorname{arcosh} 1}^{\operatorname{arcosh} 2.6} \sqrt{\cosh^2 u - 1} \cdot \frac{1}{2}\sinh u\, du$$

$$= \frac{1}{2}\int_{\operatorname{arcosh} 1}^{\operatorname{arcosh} 2.6} \sinh^2 u\, du$$

Now $\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2-1}\right)$

so $\operatorname{arcosh} 2.6 = \ln(2.6 + 2.4) = \ln 5$

and $\operatorname{arcosh} 1 = \ln(1) = 0$

Integral becomes

$$\frac{1}{2}\int_0^{\ln 5} \sinh^2 u\, du$$

which from previous part is

$$\frac{1}{2}\left(\frac{78}{25} - \frac{1}{2}\ln 5\right)$$

$$= \frac{39}{25} - \frac{1}{4}\ln 5$$

MEI PURE 5 JUNE 2004 QUESTION 2

2a)

$$\int_{-1.5}^{1.5} \frac{1}{\sqrt{9-2x^2}}\,dx$$

$$= \frac{1}{\sqrt{2}} \int_{-1.5}^{1.5} \frac{1}{\sqrt{\frac{9}{2} - x^2}}\,dx$$

$$= \frac{1}{\sqrt{2}} \int_{-1.5}^{+1.5} \frac{1}{\sqrt{\left(\frac{3}{\sqrt{2}}\right)^2 - x^2}}\,dx$$

$$= \frac{1}{\sqrt{2}} \left[\arcsin\left(\frac{x}{3/\sqrt{2}}\right) \right]_{-1.5}^{1.5}$$

$$= \frac{1}{\sqrt{2}} \left[\arcsin\left(\frac{3}{2} \times \frac{\sqrt{2}}{3}\right) - \arcsin\left(-\frac{3}{2} \times \frac{\sqrt{2}}{3}\right) \right]$$

$$= \frac{1}{\sqrt{2}} \left[\arcsin\left(\frac{1}{\sqrt{2}}\right) - \arcsin\left(-\frac{1}{\sqrt{2}}\right) \right]$$

$$= \frac{1}{\sqrt{2}} \left[\frac{\pi}{4} + \frac{\pi}{4} \right]$$

$$= \frac{\pi}{2\sqrt{2}}$$

2b) i)

$$\int_0^{\ln a} (12\cosh x - 8\sinh x)\,dx$$

$$= \int_0^{\ln a} \left(\frac{12}{2}(e^x + e^{-x}) - \frac{8}{2}(e^x - e^{-x}) \right) dx$$

$$= \int_0^{\ln a} \left(6e^x + 6e^{-x} - 4e^x + 4e^{-x} \right) dx$$

$$= \int_0^{\ln a} \left(2e^x + 10e^{-x} \right) dx$$

$$= \left[2e^x - 10e^{-x} \right]_0^{\ln a}$$

$$= \left(2e^{\ln a} - 10e^{-\ln a} \right) - \left(2e^0 - 10e^0 \right)$$

$$= 2a - \frac{10}{a} - 2 + 10$$

$$= \frac{2a^2 - 10}{a} + 8$$

ii)

$$12\cosh x - 8\sinh x = 9$$

$$2e^x + 10e^{-x} = 9$$

$$2e^{2x} + 10 = 9e^x$$

$$2e^{2x} - 9e^x + 10 = 0$$

$$\Rightarrow e^x = \frac{+9 \pm \sqrt{81-80}}{4}$$

$$e^x = \frac{+9 \pm 1}{4}$$

$$e^x = \frac{10}{4} \text{ or } \frac{8}{4}$$

$$e^x = \frac{5}{2} \text{ or } 2$$

$$x = \ln\left(\frac{5}{2}\right) \text{ or } x = \ln 2$$

MEI | PURE 5 JUNE 2004 QUESTION 2

2b) iii) Show $12\cosh x - 8\sinh x \geqslant 4\sqrt{5}$

Let $y = 12\cosh x - 8\sinh x$

and show y has min value $4\sqrt{5}$

$$y = 2e^{x} + 10e^{-x}$$

$$\frac{dy}{dx} = 2e^{x} - 10e^{-x}$$

At st. pt. $\frac{dy}{dx} = 0$

$$\Rightarrow \quad 2e^{x} - 10e^{-x} = 0$$

$$\Rightarrow \quad e^{x} - 5e^{-x} = 0$$

$$\Rightarrow \quad e^{2x} - 5 = 0$$

$$\Rightarrow \quad e^{2x} = 5$$

$$\Rightarrow \quad 2x = \ln 5$$

$$\Rightarrow \quad x = \tfrac{1}{2}\ln 5$$

$$x = \ln(\sqrt{5})$$

When $x = \ln(\sqrt{5})$

$$y = 2e^{\ln\sqrt{5}} + 10e^{-\ln\sqrt{5}}$$

$$y = 2\sqrt{5} + \frac{10}{\sqrt{5}}$$

$$y = 2\sqrt{5} + \frac{5 \times 2}{\sqrt{5}}$$

$$y = 2\sqrt{5} + 2\sqrt{5} = 4\sqrt{5}$$

$\therefore$ st. pt. at $(\ln\sqrt{5}, 4\sqrt{5})$

$$\frac{d^2y}{dx^2} = 2e^{x} + 10e^{-x}$$

When $x = \ln\sqrt{5}$

$$\frac{d^2y}{dx^2} = 2e^{\ln\sqrt{5}} + 10e^{-\ln\sqrt{5}}$$

$$= 2\sqrt{5} + \frac{10}{\sqrt{5}} > 0$$

$\therefore$ y is a minimum at $x = \ln\sqrt{5}$

$\therefore$ for all x

$$12\cosh x - 8\sinh x \geqslant 4\sqrt{5}$$

MEI PURE 5 JANUARY 2005 QUESTION 2

2a) $$\int_{2.5}^{7.5} \frac{1}{4x^2+75}\,dx$$

Let $u = 2x$

$$\frac{du}{dx} = 2$$

$$\tfrac{1}{2}du = dx$$

$x = 7.5 \Rightarrow u = 15$
$x = 2.5 \Rightarrow u = 5$

$$\frac{1}{2}\int_5^{15} \frac{1}{u^2+75}\,du$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{75}}\tan^{-1}\left(\frac{x}{\sqrt{75}}\right)\right]_5^{15}$$

$$= \frac{1}{2\sqrt{75}}\left(\tan^{-1}\left(\frac{15}{5\sqrt{3}}\right) - \tan^{-1}\left(\frac{5}{5\sqrt{3}}\right)\right)$$

$$= \frac{1}{10\sqrt{3}}\left(\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{10\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$= \frac{\pi}{60\sqrt{3}}$$

2b)i) $2\cosh^2 x - 1$

$$= 2\left(\tfrac{1}{2}(e^x + e^{-x})\right)^2 - 1$$

$$= \tfrac{1}{2}\left(e^{2x} + e^{-2x} + 2\right) - 1$$

$$= \tfrac{1}{2}\left(e^{2x} + e^{-2x}\right) + 1 - 1$$

$$= \cosh 2x$$

ii) $y = 7\sinh x - \sinh 2x$

$$\frac{dy}{dx} = 7\cosh x - 2\cosh 2x$$

At st pt. $\frac{dy}{dx} = 0$

$$\Rightarrow 7\cosh x - 2\cosh 2x = 0$$

$$7\cosh x - 2(2\cosh^2 x - 1) = 0$$

$$7\cosh x - 4\cosh^2 x + 2 = 0$$

$$4\cosh^2 x - 7\cosh x - 2 = 0$$

$$(4\cosh x + 1)(\cosh x - 2) = 0$$

Either $4\cosh x + 1 = 0$
$\cosh x = -\frac{1}{4}$ X

or $\cosh x - 2 = 0$

$$\cosh x = 2$$

$$\Rightarrow x = \pm \operatorname{arcosh} 2$$

$$x = \pm \ln(2 + \sqrt{3})$$

MEI PURE 5 JANUARY 2005 QUESTION 2

2b) ii) cont)

$$x = \pm \ln(2+\sqrt{3})$$

When $x = \ln(2+\sqrt{3})$

$$y = 7\sinh x - \sinh 2x$$

$$= \frac{7}{2}\left(e^{\ln(2+\sqrt{3})} - e^{\ln(2+\sqrt{3})}\right) - \frac{1}{2}\left(e^{2\ln(2+\sqrt{3})} - e^{2\ln(2+\sqrt{3})}\right)$$

$$= \frac{7}{2}\left(2+\sqrt{3} - \frac{1}{2+\sqrt{3}}\right) - \frac{1}{2}\left((2+\sqrt{3})^2 - \frac{1}{(2+\sqrt{3})^2}\right)$$

$$= \frac{7}{2}\left(2+\sqrt{3} - \frac{(2-\sqrt{3})}{1}\right) - \frac{1}{2}\left((2+\sqrt{3})^2 - \frac{(2-\sqrt{3})^2}{1^2}\right)$$

$$= \frac{7}{2}(2\sqrt{3}) - \frac{1}{2}(4)(2\sqrt{3})$$

$$= 7\sqrt{3} - 4\sqrt{3} = 3\sqrt{3}$$

(using difference of two squares)

When $x = -\ln(2+\sqrt{3})$

$$y = -3\sqrt{3}$$

because y is an odd fn.

2b) iii)

$$\int_0^{\ln 3} (7\sinh x - \sinh 2x)\,dx$$

$$= \int_0^{\ln 3} \left(\frac{7}{2}(e^{x} - e^{-x}) - \frac{1}{2}(e^{2x} - e^{-2x})\right)dx$$

$$= \left[\frac{7}{2}e^{x} + \frac{7}{2}e^{-x} - \frac{1}{4}e^{2x} - \frac{1}{4}e^{-2x}\right]_0^{\ln 3}$$

$$= \left(\frac{7}{2}(3) + \frac{7}{2}\left(\frac{1}{3}\right) - \frac{1}{4}(9) - \frac{1}{4}\left(\frac{1}{9}\right)\right) - \left(\frac{7}{2} + \frac{7}{2} - \frac{1}{4} - \frac{1}{4}\right)$$

$$= \left(\frac{21}{2} + \frac{7}{6} - \frac{9}{4} - \frac{1}{36}\right) - \left(7 - \frac{1}{2}\right)$$

$$= 9\tfrac{7}{18} - 6\tfrac{1}{2}$$

$$= 2\tfrac{8}{9} = \frac{26}{9}$$

MEI PURE 5 JUNE 2005 QUESTION 2

2a) i) Let $\cosh x = c$

$$\frac{1}{2}(e^{x}+e^{-x}) = c$$

$$e^{x}+e^{-x} = 2c$$

$$e^{2x}+1 = 2ce^{x}$$

$$e^{2x}-2ce^{x}+1 = 0$$

$$\Rightarrow e^{x} = \frac{+2c \pm \sqrt{4c^2-4}}{2}$$

$$\Rightarrow e^{x} = c \pm \sqrt{c^2-1}$$

$$\Rightarrow x = \ln\left(c \pm \sqrt{c^2-1}\right)$$

However the product of the roots $= \frac{c}{a} = \frac{1}{1} = 1$

$\therefore$ $c+\sqrt{c^2-1}$ and $c-\sqrt{c^2-1}$ are the reciprocals of each other. $\therefore \ln(c-\sqrt{c^2-1})$

$$= -\ln(c+\sqrt{c^2-1})$$

$$\therefore \quad x = \pm\ln\left(c+\sqrt{c^2-1}\right)$$

2a) ii) $\sinh^2 x + 3\cosh x = 9$

$$\left(\begin{array}{l}\text{Now } \cosh^2 x - \sinh^2 x = 1 \\ \therefore \cosh^2 x - 1 = \sinh^2 x\end{array}\right)$$

$$\cosh^2 x - 1 + 3\cosh x = 9$$

$$\cosh^2 x + 3\cosh x - 10 = 0$$

$$(\cosh x - 2)(\cosh x + 5) = 0$$

$$\Rightarrow \cosh x = 2 \text{ or } \cancel{\cosh x = -5}$$

$$\Rightarrow x = \operatorname{arcosh} 2$$

$$\Rightarrow x = \pm\ln(2+\sqrt{3})$$

2b) i) $f(x) = \arcsin\left(\frac{3}{5}+x\right)$

$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{3}{5}+x\right)^2}}$$

$$f'(x) = \left(1-\left(\tfrac{3}{5}+x\right)^2\right)^{-\frac{1}{2}}$$

$$f''(x) = -\frac{1}{2}\left(1-\left(\tfrac{3}{5}+x\right)^2\right)^{-3/2}\left(-2\left(\tfrac{3}{5}+x\right)\right)$$

$$f''(x) = \frac{+\left(\frac{3}{5}+x\right)}{\left(1-\left(\frac{3}{5}+x\right)^2\right)^{3/2}}$$

2b) ii) Given $f(x) = \arcsin\left(\frac{3}{5}\right) + px + qx^2 + \ldots$

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \ldots$$

$$\therefore p = f'(0) = \frac{1}{\sqrt{1-\frac{9}{25}}} = \frac{1}{\sqrt{\frac{16}{25}}}$$

$$p = \frac{5}{4}$$

MEI PURE 5 JUNE 2005 QUESTION 2

2b) ii) (cont)

$$q = \frac{f''(0)}{2}$$

$$q = \frac{\frac{3}{5}}{2\left(1-\frac{9}{25}\right)^{3/2}} = \frac{\frac{3}{5}}{2\left(\frac{16}{25}\right)^{3/2}}$$

$$q = \frac{\frac{3}{5}}{2\times\frac{64}{125}} = \frac{3}{5}\times\frac{125}{128}$$

$$q = \frac{75}{128}$$

2b) iii) $f(x) \approx \arcsin\left(\frac{3}{5}\right) + \frac{5x}{4} + \frac{75}{128}x^2$

$$\int_0^{0.1} f(x)\,dx$$

$$\approx \int_0^{0.1}\left(\arcsin\left(\frac{3}{5}\right) + \frac{5x}{4} + \frac{75}{128}x^2\right)dx$$

$$= \left[\arcsin\left(\frac{3}{5}\right)x + \frac{5x^2}{8} + \frac{25x^3}{128}\right]_0^{0.1}$$

$$= \left(0.1\arcsin\left(\frac{3}{5}\right) + \frac{5}{800} + \frac{25}{128000}\right)$$

$$-\left(0+0+0\right)$$

$$= 0.0707954$$

$$= 0.0708 \text{ to } 4 \text{ d.p.}$$