MEI STRUCTURED MATHEMATICS 5505

Pure Mathematics 5

Thursday **11 JANUARY 2001** Morning 1 hour 20 minutes

1 (a) A curve has polar equation $r = a\left(1+\sqrt{2}\cos\theta\right)$ for $-\frac{3}{4}\pi \leqslant \theta \leqslant \frac{3}{4}\pi$, where a is a positive constant.

(i) Sketch the curve. [4]

(ii) Find the area of the region enclosed by the curve. [8]

2 (a) Find $\int_0^{\frac{3}{2}} \frac{1}{\sqrt{4x^2+9}}\,dx$, giving your answer in logarithmic form. [5]

(b) (i) Sketch the graph of $y = \arccos(2x)$. [3]

(ii) Differentiate $\arccos(2x)$ with respect to x. [2]

(iii) Use integration by parts to find $\int \arccos(2x)\,dx$. [4]

(iv) By first expanding $(1-4x^2)^{-\frac{1}{2}}$, find the series expansion of $\arccos(2x)$ as far as the term in x^5. [6]

4 (i) Express $e^{jk\theta}$ and $e^{-jk\theta}$ in the form $a + jb$, and show that

$$e^{2j\theta} - 1 = 2je^{j\theta}\sin\theta.$$ [4]

Series C and S are defined by

$$C = \cos\theta + \cos 3\theta + \cos 5\theta + \ldots + \cos(2n-1)\theta,$$

$$S = \sin\theta + \sin 3\theta + \sin 5\theta + \ldots + \sin(2n-1)\theta,$$

where n is a positive integer and $0 < \theta < \frac{\pi}{n}$.

(ii) Show that $C + jS$ is a geometric series, and write down the sum of this series. [4]

(iii) Show that $|C + jS| = \frac{\sin n\theta}{\sin\theta}$, and find $\arg(C + jS)$. [5]

(iv) Find C and S. [3]

The points A_0, A_1, A_2, A_3, A_4, A_5, A_6 in the Argand diagram correspond to complex numbers z_0, z_1, z_2, z_3, z_4, z_5, z_6 where $z_0 = 0$ and $z_1 = \cos\frac{1}{7}\pi + j\sin\frac{1}{7}\pi$. The points are the vertices of a regular heptagon with sides of length 1, as shown in Fig. 4.

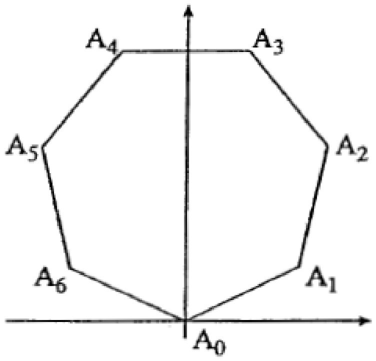
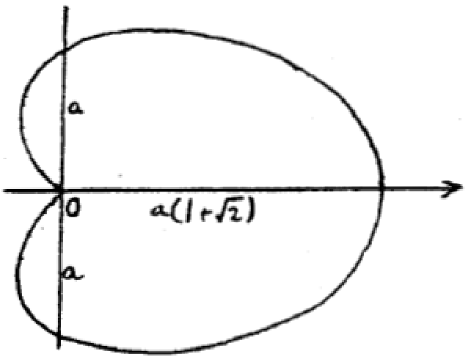
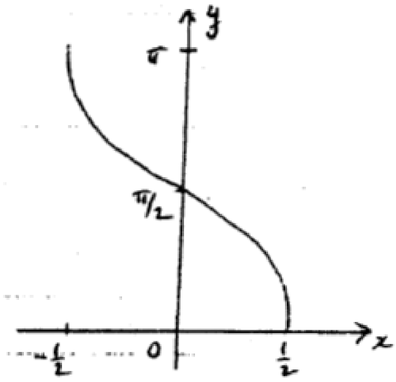


Fig. 4

(v) Explain why $z_n = e^{\frac{1}{7}j\pi} + e^{\frac{3}{7}j\pi} + \ldots + e^{\frac{1}{7}(2n-1)j\pi}$ for n = 1, 2, 3, 4, 5, 6.

Hence, or otherwise, show that $\arg(z_n) = \frac{1}{7}n\pi$ for n = 1, 2, 3, 4, 5, 6. [4]

Q	Solution	Marks	Comments
1 (a)(i)		B1 B1 B1 B1 4	Decreasing r from $\theta = 0$ to $\theta = \frac{3}{4}\pi$ Decreasing r from $\theta = 0$ to $\theta = -\frac{3}{4}\pi$ Cusp at O (and no loop inside) Indication of intercepts a and $a(1+\sqrt{2})$
2 (a)	$\int_0^{\frac{3}{2}} \frac{1}{\sqrt{4x^2+9}}\,dx = \left[\frac{1}{2}\text{arsinh}\,\frac{2x}{3}\right]_0^{\frac{3}{2}}$ $= \frac{1}{2}\text{arsinh}\,1$ $= \frac{1}{2}\ln(1+\sqrt{2})$	M1 A1 A1 M1 A1 5	For arsinh *or* any sinh substitution For arsinh $\frac{2x}{3}$ *or* $2x = 3\sinh\theta$ For factor $\frac{1}{2}$ *or* $\int \frac{1}{2}\,d\theta$ Use of $\ln(x+\sqrt{x^2+1})$
	OR $\int_0^{\frac{3}{2}} \frac{1}{\sqrt{4x^2+9}}\,dx = \left[\frac{1}{2}\ln(2x+\sqrt{4x^2+9})\right]_0^{\frac{3}{2}}$ M2 A1 A1 $= \frac{1}{2}\ln(3+\sqrt{18}) - \frac{1}{2}\ln 3$ $= \frac{1}{2}\ln(1+\sqrt{2})$ A1		Integral of form $\ln(x+\sqrt{x^2+1})$ $\ln(2x+\sqrt{4x^2+9})$ *or* $\ln(x+\sqrt{x^2+\frac{9}{4}})$ For factor $\frac{1}{2}$
(b)(i)		B1 B1 B1 3	Correct shape Indication of $(0, \frac{1}{2}\pi)$ Indication of domain $-\frac{1}{2} \le x \le \frac{1}{2}$ *Max 2 if any 'extra' included*
(ii)	$\frac{d}{dx}(\arccos 2x) = \frac{-2}{\sqrt{1-4x^2}}$	M1 A1 2	For $\frac{k}{\sqrt{1-(2x)^2}}$
(iii)	$\int \arccos 2x\,dx = x\arccos 2x + \int \frac{2x}{\sqrt{1-4x^2}}\,dx$ $= x\arccos 2x - \frac{1}{2}\sqrt{1-4x^2} + C$	M1A1 ft M1 A1 cao 4	For $\int \frac{x}{\sqrt{1-ax^2}}\,dx = k\sqrt{1-ax^2}$

(iv)	$(1-4x^2)^{-\frac{1}{2}} = 1 + 2x^2 + 6x^4 + \dots$ $\arccos 2x = -2\int (1-4x^2)^{-\frac{1}{2}}\,dx$ $= D - 2\left(x + \frac{2}{3}x^3 + \frac{6}{5}x^5 + \dots\right)$ $\arccos 0 = \frac{1}{2}\pi \Rightarrow D = \frac{1}{2}\pi$ $\arccos 2x = \frac{1}{2}\pi - 2x - \frac{4}{3}x^3 - \frac{12}{5}x^5 - \dots$	M1A1 M1 A1 ft M1 A1 cao **6**	 Integrating series *Constant not required*
4 (i)	$e^{jk\theta} = \cos k\theta + j\sin k\theta,\ e^{-jk\theta} = \cos k\theta - j\sin k\theta$ $e^{2j\theta} - 1 = e^{j\theta}(e^{j\theta} - e^{-j\theta})$ $= e^{j\theta}\{(\cos\theta + j\sin\theta) - (\cos\theta - j\sin\theta)\}$ $= 2je^{j\theta}\sin\theta$	B1 M1 M1 A1 (ag) **4**	
	OR $e^{2j\theta} - 1 = \cos 2\theta - 1 + j\sin 2\theta$ $= -2\sin^2\theta + 2j\sin\theta\cos\theta$ M1 $= 2j\sin\theta(\cos\theta + j\sin\theta)$ M1 $= 2je^{j\theta}\sin\theta$ A1		
(ii)	$C + jS = e^{j\theta} + e^{3j\theta} + e^{5j\theta} + \dots + e^{(2n-1)j\theta}$, a GP with common ratio $r = e^{2j\theta}$ $= \dfrac{e^{j\theta}(e^{2nj\theta} - 1)}{e^{2j\theta} - 1}$	M1 A1 M1A1 **4**	
(iii)	$C + jS = \dfrac{e^{j\theta}\,2je^{jn\theta}\sin n\theta}{2je^{j\theta}\sin\theta}$ $= \left(\dfrac{\sin n\theta}{\sin\theta}\right)e^{jn\theta}$ Hence $\lvert C + jS\rvert = \dfrac{\sin n\theta}{\sin\theta}$ and $\arg(C + jS) = n\theta$	M1M1 A1 A1 (ag) A1 **5**	
(iv)	$C = \dfrac{\sin n\theta\cos n\theta}{\sin\theta},\ S = \dfrac{\sin^2 n\theta}{\sin\theta}$	M1A1A1 **3**	Accept any correct real form
(v)	$\overrightarrow{A_1A_2}$ has length 1, making angle $\dfrac{3\pi}{7}$ with the real axis, so $z_2 - z_1 = e^{j\frac{3\pi}{7}}$ Similarly $z_3 - z_2 = e^{j\frac{5\pi}{7}}$, etc so $z_n = e^{j\frac{\pi}{7}} + e^{j\frac{3\pi}{7}} + \dots + e^{j\frac{(2n-1)\pi}{7}}$ $z_n = C + jS$ with $\theta = \dfrac{\pi}{7}$ Hence $\arg z_n = n\theta = \dfrac{n\pi}{7}$	 B1 B1 M1 A1 **4**	