MEI STRUCTURED MATHEMATICS 5505

Pure Mathematics 5

Tuesday **5 JUNE 2001** Morning 1 hour 20 minutes

2 **(a)** By considering $(\cos\theta + \mathrm{j}\sin\theta)^4$, express $\tan 4\theta$ in terms of $\tan\theta$. [5]

(b) **(i)** By considering $\left(z - \frac{1}{z}\right)^2\left(z + \frac{1}{z}\right)^4$, where $z = \cos\theta + \mathrm{j}\sin\theta$, show that

$$\sin^2\theta\cos^4\theta = \tfrac{1}{16} + \tfrac{1}{32}\cos 2\theta - \tfrac{1}{16}\cos 4\theta - \tfrac{1}{32}\cos 6\theta.$$ [8]

(ii) Use the substitution $x = \tan\theta$ to show that

$$\int_0^1 \frac{x^2}{\left(1 + x^2\right)^4}\,\mathrm{d}x = \frac{\pi}{64} + \frac{1}{48}.$$ [7]

3 **(i)** Given that $k \geqslant 1$ and $\cosh x = k$, prove that $x = \pm\ln\left(k + \sqrt{k^2 - 1}\right)$. [5]

In the remainder of this question, $\mathrm{f}(x) = 2\sinh^2 x - 5\cosh x$.

(ii) Solve the equation $\mathrm{f}(x) = 10$, giving your answers in an exact logarithmic form. [4]

(iii) Find the coordinates of the stationary points on the curve $y = \mathrm{f}(x)$. [6]

(iv) Show that $\displaystyle\int_0^{\ln 10} \mathrm{f}(x)\,\mathrm{d}x = \frac{99}{400} - \ln 10.$ [5]

2 (a)	$\cos 4\theta + \mathrm{j}\sin 4\theta = (\cos\theta + \mathrm{j}\sin\theta)^4$ $= \cos^4\theta + 4\cos^3\theta(\mathrm{j}\sin\theta) + 6\cos^2\theta(\mathrm{j}\sin\theta)^2 + \dots$ $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ $\tan 4\theta = \dfrac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$ $= \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$	M1 M1A1 A1 A1 cao 5	Must have binomial coefficients
(b)(i)	$z - \dfrac{1}{z} = 2\mathrm{j}\sin\theta,\quad z + \dfrac{1}{z} = 2\cos\theta$ $-64\sin^2\theta\cos^4\theta = \left(z - \dfrac{1}{z}\right)^2\left(z + \dfrac{1}{z}\right)^4$ $= z^6 + 2z^4 - z^2 - 4 - \dfrac{1}{z^2} + \dfrac{2}{z^4} + \dfrac{1}{z^6}$ $= \left(z^6 + \dfrac{1}{z^6}\right) + 2\left(z^4 + \dfrac{1}{z^4}\right) - \left(z^2 + \dfrac{1}{z^2}\right) - 4$ $= 2\cos 6\theta + 2(2\cos 4\theta) - 2\cos 2\theta - 4$ $\sin^2\theta\cos^4\theta = \frac{1}{16} + \frac{1}{32}\cos 2\theta - \frac{1}{16}\cos 4\theta - \frac{1}{32}\cos 6\theta$	B1B1 B1 M1A1 M2 A1 (ag) 8	May be implied For $-64\sin^2\theta\cos^4\theta$ If terms kept separate, award for $\cos 2\theta\cos 4\theta = \frac{1}{2}(\cos 6\theta + \cos 2\theta)$ Give M1 for one cos term (or if 2's omitted)
(ii)	$\int_0^1 \dfrac{x^2}{(1+x^2)^4}\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \dfrac{\tan^2\theta}{(\sec^2\theta)^4}\sec^2\theta\,\mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} \sin^2\theta\cos^4\theta\,\mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} \left(\frac{1}{16} + \frac{1}{32}\cos 2\theta - \frac{1}{16}\cos 4\theta - \frac{1}{32}\cos 6\theta\right)\mathrm{d}\theta$ $= \left[\frac{1}{16}\theta + \frac{1}{64}\sin 2\theta - \frac{1}{64}\sin 4\theta - \frac{1}{192}\sin 6\theta\right]_0^{\frac{\pi}{4}}$ $= \frac{\pi}{64} + \frac{1}{64} + \frac{1}{192}$ $= \frac{\pi}{64} + \frac{1}{48}$	M1 A1 A1 M1A1 M1 A1 (ag) 7	Substitution and $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = \sec^2\theta$ Limits not required Limits required (may be implied by later work)

3 (i)	$\frac{1}{2}(e^x + e^{-x}) = k$ $e^{2x} - 2ke^x + 1 = 0$ $e^x = \frac{2k \pm \sqrt{4k^2 - 4}}{2} \quad (= k \pm \sqrt{k^2 - 1})$	M1 M1 A1	($\pm$ not required)
	OR $\sinh x = \sqrt{k^2 - 1}$ (when $x > 0$) M1 $k + \sqrt{k^2 - 1} = \cosh x + \sinh x = e^x$ M1A1		
	OR $\frac{d}{dk}\ln(k + \sqrt{k^2 - 1}) = ... = \frac{1}{\sqrt{k^2 - 1}}$ B2 $\ln(k + \sqrt{k^2 - 1}) = \cosh^{-1} k + C$ When $k = 1$, $0 = 0 + C$, so $C = 0$ B1		
	$(k - \sqrt{k^2 - 1})(k + \sqrt{k^2 - 1}) = k^2 - (k^2 - 1) = 1$ so $k - \sqrt{k^2 - 1} = \frac{1}{k + \sqrt{k^2 - 1}}$ $x = \ln(k + \sqrt{k^2 - 1})$ or $x = \ln\left(\frac{1}{k + \sqrt{k^2 - 1}}\right)$ $x = \pm\ln(k + \sqrt{k^2 - 1})$	M1 A1 (ag) 5	*or* since $\cosh(-x) = \cosh x$, if $x = \lambda$ is one solution, the other must be $x = -\lambda$ *or* (when $x < 0$), $\sinh x = -\sqrt{k^2 - 1}$, $k + \sqrt{k^2 - 1} = \cosh x - \sinh x = e^{-x}$
(ii)	$2(\cosh^2 x - 1) - 5\cosh x = 10$ $2\cosh^2 x - 5\cosh x - 12 = 0$ $(\cosh x - 4)(2\cosh x + 3) = 0$ $\cosh x = 4$ $x = \pm\ln(4 + \sqrt{15})$	M1 M1 A1 A1 cao 4	*Dependent on previous M1* *Ignore* $\cosh x = -\frac{3}{2}$ *if stated* *or* $x = \ln(4 \pm \sqrt{15})$ Give A0 if any other solutions stated
(iii)	$f'(x) = 4\sinh x\cosh x - 5\sinh x$ $= \sinh x(4\cosh x - 5)$ $f'(x) = 0$ when $\sinh x = 0$, $\cosh x = \frac{5}{4}$ $x = 0$, $x = \pm\ln 2$ Stationary points $(0, -5)$ $\left(\ln 2, -\frac{41}{8}\right)$ $\left(-\ln 2, -\frac{41}{8}\right)$	M1A1 M1 A1 A2 6	One term correct is sufficient for M1 *Accept* 0.69 *or* $\cosh^{-1}\frac{5}{4}$ *for* x *but* $y = -5.125$ *must be exact* Give A1 for one correct
(iv)	$\int_0^{\ln 10} f(x)\,dx = \int_0^{\ln 10} (\cosh 2x - 1 - 5\cosh x)\,dx$ $= \left[\frac{1}{2}\sinh 2x - x - 5\sinh x\right]_0^{\ln 10}$ $= \frac{1}{2}\sinh(2\ln 10) - \ln 10 - 5\sinh(\ln 10)$ $= \frac{1}{4}\left(100 - \frac{1}{100}\right) - \ln 10 - \frac{5}{2}\left(10 - \frac{1}{10}\right)$ $= \frac{99}{400} - \ln 10$	M1 M1A1 M1 A1 (ag) 5	Exact evaluation of $\sinh(\ln 10)$ *or* $\sinh(2\ln 10)$ *Do not accept (e.g.)* $\sinh(\ln 10) = 4.95$ *without any working*