OCR RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2605

Pure Mathematics 5

Wednesday **12 JANUARY 2005** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 3 printed pages and 1 blank page.

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[Turn over

1 (a) The equation $8x^4 + 16x^3 + 1 = 0$ has roots α, β, γ and δ.

Use a suitable substitution to find a quartic equation with integer coefficients which has roots

$$8\alpha^3, 8\beta^3, 8\gamma^3 \text{ and } 8\delta^3.$$ [5]

(b) When the polynomial $f(x) = x^5 + kx^4 + mx^3 + 7x - 2$ is divided by $(x-2)$, the remainder is 12.

When $f(x)$ is divided by $(x+1)$, the remainder is 3.

(i) Find k and m. [4]

(ii) Find the remainder when $f(x)$ is divided by $(x-2)(x+1)$. [4]

(iii) Show that $f'(-1) = -30$. [2]

(iv) Find the remainder when $f(x)$ is divided by $(x+1)^2$. [5]

2 (a) Find the exact value of $\int_{2.5}^{7.5} \frac{1}{4x^2+75}\,dx$. [5]

(b) (i) Starting from $\cosh x = \frac{1}{2}(e^x + e^{-x})$, show that $\cosh 2x = 2\cosh^2 x - 1$. [3]

(ii) Show that the two stationary points on the curve $y = 7\sinh x - \sinh 2x$ have y-coordinates $3\sqrt{3}$ and $-3\sqrt{3}$. [7]

(iii) Show that $\int_0^{\ln 3} (7\sinh x - \sinh 2x)\,dx = \frac{26}{9}$. [5]

3 In this question, $z = \cos\theta + j\sin\theta$ where θ is real.

(a) By considering z^5, express $\tan 5\theta$ in terms of $\tan\theta$. [6]

(b) (i) Write $z^n + \frac{1}{z^n}$ and $z^n - \frac{1}{z^n}$ in simplified trigonometric form. [3]

(ii) By considering $\left[\left(z-\frac{1}{z}\right)\left(z+\frac{1}{z}\right)\right]^3$, show that $\sin^3\theta\cos^3\theta = \frac{3}{32}\sin 2\theta - \frac{1}{32}\sin 6\theta$. [6]

(iii) Hence find the first two non-zero terms of the Maclaurin series for $\sin^3\theta\cos^3\theta$. [3]

(iv) Given that θ^7 and higher powers may be neglected, show that

$$\sin^3\theta\cos^3\theta \approx \theta^3\cos 2\theta.$$ [2]

4 (a) A curve has polar equation $r = k\sin 4\theta$, for $0 \leqslant \theta \leqslant \pi$, where k is a positive constant.

(i) Sketch the curve, using a continuous line for sections where $r > 0$, and a broken line for sections where $r < 0$. [3]

(ii) Find the area of one loop of the curve. [5]

(b) $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The tangents to the hyperbola at P_1 and P_2 meet at the point $Q(r, s)$.

(i) Find (in terms of a, b, x_1 and y_1) the gradient of the hyperbola at P_1, and hence show that the equation of the tangent at P_1 is $\frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = 1$. [5]

(ii) Show that P_1 and P_2 lie on the line with equation $\frac{rx}{a^2} - \frac{sy}{b^2} = 1$. [3]

(iii) Given that Q lies on a directrix of the hyperbola, show that the line P_1P_2 passes through a focus. [4]

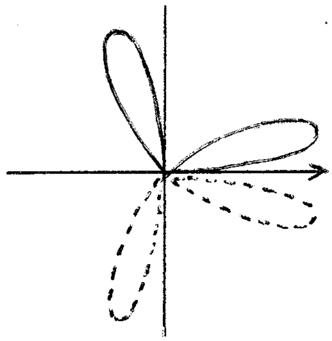
Mark Scheme

MARKING SCHEME

1 (a)	Let $y = 8x^3$, $x = \frac{1}{2}y^{\frac{1}{3}}$ $\frac{1}{2}y^{\frac{4}{3}} + 2y + 1 = 0$ $-\frac{1}{8}y^4 = (2y+1)^3$ $y^4 + 64y^3 + 96y^2 + 48y + 8 = 0$	M1 A1 M1 A1 A1 **5**	Obtaining equation in *y* Eliminating fractional powers *SR* Give B3 for correct answer obtained by another method
(b)(i)	$f(2) = 12 \Rightarrow 32 + 16k + 8m + 14 - 2 = 12$ $(16k + 8m = -32)$ $f(-1) = 3 \Rightarrow -1 + k - m - 7 - 2 = 3$ $(k - m = 13)$ $k = 3$, $m = -10$	M1 A1 M1 A1 **4**	Substituting $x = 2$ *or* $x = -1$ (or long division leading to a remainder in terms of *k* and *m*) Both equations correct Solving to obtain *k* or *m*
(ii)	$f(x) = (x-2)(x+1)g(x) + ax + b$ Putting $x = 2$, $12 = 2a + b$ Putting $x = -1$, $3 = -a + b$ $a = 3$, $b = 6$ Remainder is $3x + 6$	B1 M1 A1 A1 **4**	May be implied Substituting $x = 2$ *or* $x = -1$ Both equations correct
	OR by long division Obtaining quotient M1 Quotient is $x^3 + 4x^2 - 4x + 4$ A1 Obtaining a linear remainder M1 Remainder is $3x + 6$ A1		All four terms required
(iii)	$f'(x) = 5x^4 + 12x^3 - 30x^2 + 7$ $f'(-1) = 5 - 12 - 30 + 7 = -30$	M1 A1 (ag) **2**	
(iv)	$f(x) = (x+1)^2 h(x) + px + q$ $f'(x) = (x+1)^2 h'(x) + 2(x+1)h(x) + p$ Putting $x = -1$, $3 = -p + q$ $-30 = p$ Remainder is $-30x - 27$	B1 M1 B1 A1 B1 **5**	
	OR by long division Obtaining quotient M1 Quotient is $x^3 + x^2 - 13x + 25$ A2 Obtaining a linear remainder M1 Remainder is $-30x - 27$ A1		All four terms required Give A1 for $x^3 + x^2 + \ldots$

2 (a)	$\left[\frac{1}{2}\frac{1}{\sqrt{75}}\arctan\left(\frac{2x}{\sqrt{75}}\right)\right]_{2.5}^{7.5}$	M1 A1A1	For arctan For $\frac{1}{2\sqrt{75}}$ and $\frac{2x}{\sqrt{75}}$
	$=\frac{1}{2\sqrt{75}}\left(\arctan\sqrt{3}-\arctan\frac{1}{\sqrt{3}}\right)$ $=\frac{1}{2\sqrt{75}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)$	M1	Exact evaluation of $\arctan\sqrt{3}$ or $\arctan\frac{1}{\sqrt{3}}$
	$=\frac{\pi}{12\sqrt{75}}$ $\quad\left(=\frac{\pi}{60\sqrt{3}}\right)$	A1 **5**	
	OR Let $2x=\sqrt{75}\tan\theta$ M1 Integral is $\int\frac{1}{2\sqrt{75}}\,\mathrm{d}\theta$ A1 Limits are $\frac{1}{6}\pi$ and $\frac{1}{3}\pi$ A1 Integral is $\frac{\pi}{12\sqrt{75}}$ M1 A1		Any tan substitution For either
(b)(i)	$\text{RHS}=2[\frac{1}{2}(e^x+e^{-x})]^2-1$ $=\frac{1}{2}(e^{2x}+2+e^{-2x})-1$ $=\frac{1}{2}(e^{2x}+e^{-2x})$ $=\cosh 2x$	B1 B1 B1 (ag) **3**	For $(e^x+e^{-x})^2=e^{2x}+2+e^{-2x}$ For $\frac{1}{2}(e^{2x}+e^{-2x})=\cosh 2x$ For completion
(ii)	$\frac{dy}{dx}=7\cosh x-2\cosh 2x$ $=0$ when $7\cosh x-2(2\cosh^2 x-1)=0$ $4\cosh^2 x-7\cosh x-2=0$ $\cosh x=2$ Two stationary points, at $x=\pm\operatorname{arcosh}2$ $\sinh x=\pm\sqrt{\cosh^2 x-1}=\pm\sqrt{3}$ $y=7\sinh x-2\sinh x\cosh x$ $=\sinh x(7-2\cosh x)$ $=\pm\sqrt{3}(7-4)$ $=\pm 3\sqrt{3}$	B1 M1 A1 M1 M1 M1 A1 (ag) **7**	 Using (i) *(No need to reject $-\frac{1}{4}$ explicitly)* Two values (may be implied) $\pm$ not required. M0 if not exact For $\sinh 2x=(\pm)\,4\sqrt{3}$ $\pm$ not required. M0 if not exact
(iii)	$\left[7\cosh x-\frac{1}{2}\cosh 2x\right]_0^{\ln 3}$ $=\frac{7}{2}\left(3+\frac{1}{3}\right)-\frac{1}{4}\left(9+\frac{1}{9}\right)-\left(7-\frac{1}{2}\right)$ $=\frac{26}{9}$	B1B1 M1 M1 A1 (ag) **5**	For $7\cosh x$ and $-\frac{1}{2}\cosh 2x$ Exact evaluation of $\cosh(\ln 3)$ *or* $\cosh(2\ln 3)$ For $-(7-\frac{1}{2})$ Exact value correctly obtained

3(a)	$\cos 5\theta + \mathrm{j}\sin 5\theta = (\cos\theta + \mathrm{j}\sin\theta)^5$ $= c^5 + 5\mathrm{j}c^4 s - 10c^3 s^2 - 10\mathrm{j}c^2 s^3 + 5cs^4 + \mathrm{j}s^5$ $\tan 5\theta = \dfrac{\sin 5\theta}{\cos 5\theta} = \dfrac{5c^4 s - 10c^2 s^3 + s^5}{c^5 - 10c^3 s^2 + 5cs^4}$ $= \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$	M1 A1 M1 A1 M1 A1 **6**	For statement of deMoivre's theorem *or* expanding $(c + \mathrm{j}s)^5$ Equating real or imaginary parts $\sin 5\theta$ and $\cos 5\theta$ both correct Writing in terms of $\tan\theta$
(b)(i)	$z^n + \dfrac{1}{z^n} = 2\cos n\theta$ $z^n - \dfrac{1}{z^n} = 2\mathrm{j}\sin n\theta$	M1 A1 A1 **3**	For $z^n = \cos n\theta + \mathrm{j}\sin n\theta$
(ii)	$\left[\left(z - \dfrac{1}{z}\right)\left(z + \dfrac{1}{z}\right)\right]^3 = -64\mathrm{j}\sin^3\theta\cos^3\theta$ $\left(z^2 - \dfrac{1}{z^2}\right)^3 = z^6 - 3z^2 + \dfrac{3}{z^2} - \dfrac{1}{z^6}$ $= 2\mathrm{j}\sin 6\theta - 6\mathrm{j}\sin 2\theta$ $\sin^3\theta\cos^3\theta = \frac{3}{32}\sin 2\theta - \frac{1}{32}\sin 6\theta$	B1 M1A1 M1A1 A1 (ag) **6**	
(iii)	$\dfrac{3}{32}\left(2\theta - \dfrac{(2\theta)^3}{3!} + \dfrac{(2\theta)^5}{5!} - \ldots\right)$ $- \dfrac{1}{32}\left(6\theta - \dfrac{(6\theta)^3}{3!} + \dfrac{(6\theta)^5}{5!} - \ldots\right)$ $= \theta^3 - 2\theta^5 + \ldots$	B1 B1 B1 **3**	Expansion of $\sin 2\theta$ Expansion of $\sin 6\theta$ If B0, give B1 for both expansions correct up to θ^3
	OR $\mathrm{f}(\theta) = \frac{3}{32}\sin 2\theta - \frac{1}{32}\sin 6\theta$ $\mathrm{f}^{(3)}(0) = 6$, $\mathrm{f}^{(5)}(0) = -240$ B1B1 $\mathrm{f}(\theta) = \theta^3 - 2\theta^5 + \ldots$ B1		Accept coefficients in any correct form
(iv)	$\theta^3\cos 2\theta = \theta^3\left(1 - \dfrac{(2\theta)^2}{2!} + \ldots\right)$ $= \theta^3 - 2\theta^5 + \ldots$ Hence $\sin^3\theta\cos^3\theta \approx \theta^3\cos 2\theta$	M1 A1 **2**	Using expansion of $\cos 2\theta$ For completion

4(a)(i)		B1 B1 B1 **3**	2 loops correct 4 loops correct Fully correct, continuous and broken lines, and no extra loops
(ii)	Area is $\int_0^{\frac{\pi}{4}} \frac{1}{2}(k\sin 4\theta)^2\,\mathrm{d}\theta$ $=\frac{1}{4}k^2\int_0^{\frac{\pi}{4}}(1-\cos 8\theta)\,\mathrm{d}\theta \quad =\frac{1}{4}k^2\left[\ \theta-\frac{1}{8}\sin 8\theta\ \right]_0^{\frac{\pi}{4}}$ $=\frac{1}{16}\pi k^2$	M1 A1 B1B1 A1 **5**	For integral of $\sin^2 4\theta$ Correct integral form For $\int\sin^2 4\theta\,\mathrm{d}\theta=\frac{1}{2}\theta-\frac{1}{16}\sin 8\theta$ Accept $0.196k^2$
(b)(i)	$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x}=0$ Gradient at P_1 is $\frac{b^2x_1}{a^2y_1}$ Tangent is $y-y_1=\frac{b^2x_1}{a^2y_1}(x-x_1)$ $\frac{x_1x}{a^2}-\frac{y_1y}{b^2}=\frac{x_1^{\,2}}{a^2}-\frac{y_1^{\,2}}{b^2}$ $\frac{x_1x}{a^2}-\frac{y_1y}{b^2}=1$	M1 A1 M1 M1 A1 (ag) **5**	Using $\frac{x_1^{\,2}}{a^2}-\frac{y_1^{\,2}}{b^2}=1$
(ii)	Q lies on tangent at P_1 so $\frac{x_1r}{a^2}-\frac{y_1s}{b^2}=1$ Tangent at P_2 is $\frac{x_2x}{a^2}-\frac{y_2y}{b^2}=1$ Q lies on tangent at P_2 so $\frac{x_2r}{a^2}-\frac{y_2s}{b^2}=1$ Hence $P_1(x_1\,,\ y_1)$ and $P_2(x_2\,,\ y_2)$ both lie on the line $\frac{rx}{a^2}-\frac{sy}{b^2}=1$	B1 B1 B1 (ag) **3**	For completion
(iii)	Q lies on directrix $\Leftrightarrow\ r=\pm\frac{a}{e}$ P_1P_2 passes through focus $(\pm ae,\ 0)$ if and only if $\pm\frac{rae}{a^2}-0=1,$ i.e. $r=\pm\frac{a}{e}$ Hence, if Q lies on a directrix, P_1P_2 passes through a focus	B1 M1 A1 A1 **4**	$\pm$ not required Using equation of P_1P_2 $\pm$ not required Must consider both directrices & foci

Examiner’s Report