FP2 PURE 5 JANUARY 2003 Q2 COMPLEX NUMBERS

2 a) By de Moivre's theorem

$$\cos 5\theta + j\sin 5\theta = (\cos\theta + j\sin\theta)^5$$

Let $c = \cos\theta$, $s = \sin\theta$

$$\cos 5\theta + j\sin 5\theta = c^5 + 5c^4js + 10c^3j^2s^2 + 10c^2j^3s^3 + 5cj^4s^4 + j^5s^5$$

$$= c^5 + 5c^4js - 10c^3s^2 - 10c^2js^3 + 5cs^4 + js^5$$

Equating imaginary parts

$$\sin 5\theta = 5c^4s - 10c^2s^3 + s^5$$

$$= 5(1-s^2)^2s - 10(1-s^2)s^3 + s^5$$

$$= 5(1-2s^2+s^4)s - 10(s^3-s^5) + s^5$$

$$= 5s - 10s^3 + 5s^5 - 10s^3 + 10s^5 + s^5$$

$$= 5s - 20s^3 + 16s^5$$

$$\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$$

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b) i) $\left(z + \frac{1}{z}\right)^6 = (2\cos\theta)^6 = 64\cos^6\theta$

Also $\left(z + \frac{1}{z}\right)^6 = z^6 + 6z^5\frac{1}{z} + 15z^4\frac{1}{z^2} + 20z^3\frac{1}{z^3} + 15z^2\frac{1}{z^4} + 6z\frac{1}{z^5} + \frac{1}{z^6}$

$$= \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$$

FP 2 PURE 5 JANUARY 2003 Q2 COMPLEX NUMBERS

2bi) cont)

$$\Rightarrow 64\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$$

$$\Rightarrow \cos^6\theta = \frac{\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10}{32}$$

$$= \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$$

2bii)

$$\int_0^{\frac{1}{2}} (1-x^2)^{5/2}\,dx$$

Let $x = \sin\theta$

$$\frac{dx}{d\theta} = \cos\theta$$

$$dx = \cos\theta\,d\theta$$

when $x = \frac{1}{2}$ $\theta = \frac{\pi}{6}$

when $x = 0$ $\theta = 0$

$$= \int_0^{\frac{\pi}{6}} (1-\sin^2\theta)^{5/2}\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} (\cos^2\theta)^{5/2}\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} \cos^6\theta\,d\theta = \int_0^{\frac{\pi}{6}} \left(\frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}\right)d\theta$$

$$= \left[\frac{1}{192}\sin 6\theta + \frac{3}{64}\sin 4\theta + \frac{15}{64}\sin 2\theta + \frac{5}{16}\theta\right]_0^{\frac{\pi}{6}}$$

$$= \left[\frac{1}{192}\sin\pi + \frac{3}{64}\sin\frac{2\pi}{3} + \frac{15}{64}\sin\frac{\pi}{3} + \frac{5}{96}\right] - \left[0\right]$$

$$= \frac{3}{64}\times\frac{\sqrt{3}}{2} + \frac{15}{64}\times\frac{\sqrt{3}}{2} + \frac{5}{96}$$

$$= \frac{3\sqrt{3}}{128} + \frac{15\sqrt{3}}{128} + \frac{5}{96} = \frac{18\sqrt{3}}{128} + \frac{5}{96} = \frac{9\sqrt{3}}{64} + \frac{5}{96}$$