2)
a) By de Moivre's theorem

$$(\cos\theta + j\sin\theta)^4 = \cos 4\theta + j\sin 4\theta$$

Let $c = \cos\theta$, $s = \sin\theta$

$$\Rightarrow \cos 4\theta + j\sin 4\theta = (c + js)^4$$

$$= c^4 + 4c^3js + 6c^2j^2s^2 + 4cj^3s^3 + j^4s^4$$

$$= c^4 + 4c^3js - 6c^2s^2 - 4cjs^3 + s^4$$

Equating real and imaginary parts

$$\cos 4\theta = c^4 - 6c^2s^2 + s^4$$

$$\sin 4\theta = 4c^3s - 4cs^3$$

$$\Rightarrow \tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta} = \frac{4c^3s - 4cs^3}{c^4 - 6c^2s^2 + s^4}$$

Dividing numerator and denominator by c^4 gives

$$\tan 4\theta = \frac{\frac{4c^3s}{c^4} - \frac{4cs^3}{c^4}}{\frac{c^4}{c^4} - \frac{6c^2s^2}{c^4} + \frac{s^4}{c^4}}$$

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

2b) i) Let $z = \cos\theta + j\sin\theta$

$$\Rightarrow \quad z + \frac{1}{z} = 2\cos\theta \quad , \quad z - \frac{1}{z} = 2j\sin\theta$$

$$\Rightarrow \quad \left(z - \frac{1}{z}\right)^2\left(z + \frac{1}{z}\right)^4 = \left(2j\sin\theta\right)^2\left(2\cos\theta\right)^4$$

$$= -4\sin^2\theta \times 16\cos^4\theta$$

$$= -64\sin^2\theta\cos^4\theta$$

However,

$$\left(z - \frac{1}{z}\right)^2\left(z + \frac{1}{z}\right)^4 = \left(z^2 - 2 + \frac{1}{z^2}\right)\left(z^4 + 4z^3\frac{1}{z} + 6z^2\frac{1}{z^2} + 4z\frac{1}{z^3} + \frac{1}{z^4}\right)$$

$$= \left(z^2 - 2 + \frac{1}{z^2}\right)\left(z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}\right)$$

$$\begin{aligned} = z^6 - 2z^4 &+ z^2 \\ +4z^4 &- 8z^2 + 4 \\ &+ 6z^2 - 12 + \frac{6}{z^2} \\ &\qquad + 4 - \frac{8}{z^2} + \frac{4}{z^4} \\ &\qquad\qquad + \frac{1}{z^2} - \frac{2}{z^4} + \frac{1}{z^6} \end{aligned}$$

$$= z^6 + 2z^4 - z^2 - 4 - \frac{1}{z^2} + \frac{2}{z^4} + \frac{1}{z^6}$$

$$= \left(z^6 + \frac{1}{z^6}\right) + 2\left(z^4 + \frac{1}{z^4}\right) - 1\left(z^2 + \frac{1}{z^2}\right) - 4$$

$$= 2\cos 6\theta + 4\cos 4\theta - 2\cos 2\theta - 4$$

2bi) cont)

$$\therefore \quad -64\sin^2\theta\cos^4\theta = 2\cos 6\theta + 4\cos 4\theta - 2\cos 2\theta - 4$$

$$\Rightarrow \quad \sin^2\theta\cos^4\theta = \frac{2\cos 6\theta + 4\cos 4\theta - 2\cos 2\theta - 4}{-64}$$

$$\Rightarrow \quad \sin^2\theta\cos^4\theta = \frac{1}{16} - \frac{1}{32}\cos 6\theta - \frac{1}{16}\cos 4\theta + \frac{1}{32}\cos 2\theta$$

2bii)

$$\int_0^1 \frac{x^2}{(1+x^2)^4}\,dx$$

Let $x = \tan\theta$

$$\frac{dx}{d\theta} = \sec^2\theta$$

$$dx = \sec^2\theta\,d\theta$$

When $x = 1$, $\theta = \frac{\pi}{4}$

when $x = 0$, $\theta = 0$

$$= \int_0^{\frac{\pi}{4}} \frac{\tan^2\theta\sec^2\theta\,d\theta}{(1+\tan^2\theta)^4}$$

$$= \int_0^{\frac{\pi}{4}} \frac{\tan^2\theta\sec^2\theta\,d\theta}{\sec^8\theta}$$

$$= \int_0^{\frac{\pi}{4}} \frac{\tan^2\theta}{\sec^6\theta}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{\sin^2\theta}{\cos^2\theta} \times \cos^6\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sin^2\theta\cos^4\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{16} - \frac{1}{32}\cos 6\theta - \frac{1}{16}\cos 4\theta + \frac{1}{32}\cos 2\theta\right) d\theta$$

$$= \left[\frac{\theta}{16} - \frac{1}{192}\sin 6\theta - \frac{1}{64}\sin 4\theta + \frac{1}{64}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{\pi}{64} + \frac{1}{192} - 0 + \frac{1}{64}\right) - (0)$$

$$= \frac{\pi}{64} + \frac{1}{48}$$