FP2 PURE 5 MAY 2002 Q2 COMPLEX NUMBERS

2 i) $0 < k < 1 \qquad 0 < \theta < \frac{\pi}{2}$

$$e^{j\theta} + e^{-j\theta} = (\cos\theta + j\sin\theta) + (\cos\theta - j\sin\theta) = 2\cos\theta$$

$$\begin{aligned}(1-ke^{j\theta})(1-ke^{-j\theta}) &= 1 - ke^{j\theta} - ke^{-j\theta} + k^2 \\ &= 1 + k^2 - k(e^{j\theta} + e^{-j\theta}) \\ &= 1 + k^2 - 2k\cos\theta\end{aligned}$$

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ii)

$$C = k\cos\theta + k^2\cos 2\theta + k^3\cos 3\theta + \dots$$

$$S = k\sin\theta + k^2\sin 2\theta + k^3\sin 3\theta + \dots$$

$$\begin{aligned}C + jS &= k(\cos\theta + j\sin\theta) + k^2(\cos 2\theta + j\sin 2\theta) + \dots \\ &= ke^{j\theta} + k^2e^{j2\theta} + k^3e^{j3\theta} + \dots\end{aligned}$$

This is a GP with first term $a = ke^{j\theta}$

common ratio $r = ke^{j\theta}$

This is an infinite GP with $|r| = k < 1$ $\therefore$ its sum exists

iii)

$$S_\infty = \frac{a}{1-r} = \frac{ke^{j\theta}}{1-ke^{j\theta}}$$

$$= \frac{ke^{j\theta}}{1-ke^{j\theta}} \times \frac{1-ke^{-j\theta}}{1-ke^{-j\theta}}$$

$$= \frac{ke^{j\theta} - k^2}{1 - ke^{j\theta} - ke^{-j\theta} + k^2}$$

$$= \frac{ke^{j\theta} - k^2}{1 + k^2 - 2k\cos\theta}$$

2 iii) cont)

$$= \frac{k(\cos\theta + j\sin\theta) - k^2}{1 + k^2 - 2k\cos\theta}$$

Equating real and imaginary parts

$$C = \frac{k\cos\theta - k^2}{1 - 2k\cos\theta + k^2}$$

$$S = \frac{k\sin\theta}{1 - 2k\cos\theta + k^2}$$

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iv) Given that $C = 0$

$$\Rightarrow \quad k\cos\theta - k^2 = 0$$

$$\Rightarrow \quad \cos\theta = k$$

$$\Rightarrow \quad S = \frac{k\sqrt{1-k^2}}{1 - 2k^2 + k^2}$$ since $\sin\theta = \sqrt{1-\cos^2\theta}$

$$\Rightarrow \quad S = \frac{k\sqrt{1-k^2}}{1-k^2}$$

$$\Rightarrow \quad S = \frac{k}{\sqrt{1-k^2}}$$

$$\Rightarrow \quad S = \frac{\cos\theta}{\sin\theta} = \cot\theta$$
